UDC 532.5

AVERAGING OF TRANSPORT EQUATIONS WITH RESPECT TO ANGLES IN THE TWO-DIMENSIONAL CASE

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A method is proposed for solving problems of unsteady gasdynamics with radiation heat exchange taken into consideration, which do not require the use of angular distribution for each interval of time. The system of radiation transport equations averaged with respect to angles is derived for the case of cylindrical symmetry. This system is equivalent to the basic system of transport equations, in that it yields the same density for the radiation stream at the instant of averaging as the latter. Similar one-dimensional problems were considered in [1–4]. Solution of unsteady gasdynamic problems with allowance for radiation heat exchange involves a considerable amount of calculations because the transport equation which defines the radiation field has to be formulated for spectral intensity propagating in a determined direction is defined by two angles. At the same time, the equations of gasdynamics contain only the integral characteristic of the radiation field, namely, the radiation stream density.

Averaged equations are derived below for the case in which parameters of the substance are constant at cylindrical surfaces with a common axis of symmetry.

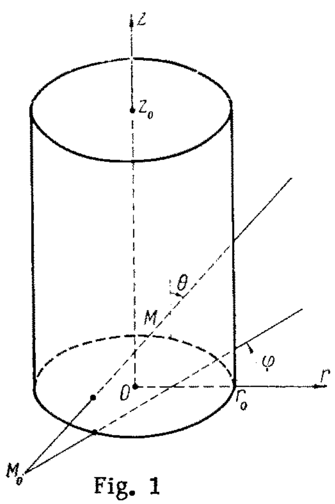

Fig. 1

In the case of cylindrical symmetry, neglecting dissipation and the time of radiation propagation over a heated and cooled volume, the equation of radiation transport is of the form

$$\sin\theta\cos\varphi\frac{\partial I}{\partial r}-\frac{1}{r}\frac{\partial I}{\partial\varphi}\sin\theta\sin\varphi+\frac{\partial I}{\partial z}\cos\theta=-K(I-B) \qquad (1)$$

where r, z and ψ are cylindrical coordinates of point M (Fig. 1), θ is the angle between ray M_0M and the z-axis, φ is the angle between the projection of ray M_0M and that of the position vector of point M on the plane $z=0$, M_0 is the intersection point of a ray with the plane $z=0$, I is the radiation intensity multiplied by π and related to a unit region of quantum energy ε, K is the linear spectral absorption coefficient corrected for forced emission, and B is the Plank function.

The averaged equations of the stream radiation density $\mathbf{q}$ are

$$\mathbf{q} = \int_{\Omega} \mathbf{I} d\Omega, \qquad \mathbf{q} = \{q_r, q_z, 0\} \tag{2}$$

where Ω is a sphere of unit radius with its center at point $M(r, z, \psi)$, the direction of $\mathbf{I}(r, z, \varphi, \theta)$ is defined by (φ, θ) , and $|\mathbf{I}(r, z, \varphi, \theta)| = I(r, z, \varphi, \theta)$.

For the sake of simplicity let us consider the problem for a straight cylinder in which function I is determined in region

$$G = \{r \in (0, r_0),\ z \in (0, z_0),\ \varphi \in (0, 2\pi),\ \theta \in (0, \pi)\}$$

where r_0 and z_0 are dimensions of the cylinder (Fig. 1).

First, let us examine q_r. We divide q_r into q_r^+ and q_r^-

$$q_r = q_r^+ + q_r^-, \qquad q_r^{\pm} = \int_{\sigma^{\pm}} I \sin\theta \cos\varphi \, d\Omega$$

$$\sigma^+ = \{\theta \in (0, \pi),\ \varphi \in (-\pi/2, \pi/2)\},\ \sigma^- = \{\theta \in (0, \pi),\ \varphi \in (\pi/2, {}^3/_2\,\pi)\}$$

We introduce in G the function $\psi_r^{\pm} = I / q_r^{\pm}$, where $q_r^{\pm}$ is determined in region $D = \{r \in (0, r_0),\ z \in (0, z_0)\}$. Substituting $I = q_r^{\pm}\psi_r^{\pm}$ into (1) and integrating over the hemispheres $\sigma^{\pm}$, we obtain two equations of the hyperbolic kind

$$\partial q_r^{\pm} / \partial r + c_{12}^{\pm} \partial q_r^{\pm} / \partial z + c_1^{\pm} q_r^{\pm} = 2\pi B \tag{3}$$

where

$$c_{12}^{\pm} = \int_{\sigma^{\pm}} \psi_r^{\pm} \cos\theta \, d\Omega, \qquad c_1^{\pm} = \frac{\partial c_{12}^{\pm}}{\partial z} - \frac{1}{r} \int_{\sigma^{\pm}} \frac{\partial \psi_r^{\pm}}{\partial \varphi} \sin\theta \sin\varphi \, d\Omega + K \int_{\sigma^{\pm}} \psi_r^{\pm} d\Omega$$

The equations of characteristics are

$$dz^{\pm} / dr = c_{12}^{\pm} \tag{4}$$

and the boundary conditions are

$$q_r^-(r_0, z) = 0,\ \ q_r^+(0, z) = -q_r^-(0, z) \tag{5}$$

Thus the subdivision of q_r into q_r^+ and q_r^- has made it possible to obtain a system of two independent equations for $q_r^{\pm}$ and their boundary conditions.

System (3) is determined in D and the boundary conditions (5) are specified at Γ (the part $r = r_0, r = 0$ of the boundary D). Let us prove that system (3) with boundary conditions (5) can be further defined and solved in $D \cup \Gamma$. To do this we shall, first, prove the following statements.

1. A solution of Eq. (4) with initial conditions

$$z(r)|_{r=r_0} = z_*,\ \ z_* \in [\delta, z_0 - \delta]\ \ (\delta > 0) \tag{6}$$

exists and is unique.

2. The solution of Eq. (4) with initial conditions (6) fills the entire region D.
3. A solution of equation

$$dq_r^{\pm} / dr = f^{\pm}(q_r^{\pm}, r),\ \ f^{\pm} = -c_1^{\pm}(r, z^{\pm}(r))\, q_r^{\pm} + 2\pi B \tag{7}$$

with boundary conditions

$$q_r^-|_{r=r_0} = 0, \qquad q_r^+|_{r=0} = -q_r^-|_{r=0} \tag{8}$$

exists and is unique.

Let us prove statements 1–3 for a uniformly heated cylinder in the absence of external radiation sources. In this case

$$I(S) = B(1 - e^{-Ks})$$

$$\lim_{r\to r_0} c_{12}^- = 0, \qquad \lim_{r\to 0} c_{12}^+ = -\lim_{r\to 0} c_{12}^- < \infty$$

$$|\partial c_{12}^- / \partial z| \leqslant \text{const}\,(r_0 - r)^{-1/2} \text{ в } d_0 \tag{9}$$

$$d_0 = \{z \in [\delta, z_0 - \delta],\ r \in [r_0 - \delta_0, r_0)\}\ \delta > 0,\ \delta_0 > 0$$

where s is the distance between M and M_0 (Fig. 1). The continuity of $c_{12}^{\pm}$ and $\partial c_{12}^{\pm}/\partial z$ in D, and the properties of (9) imply that the conditions of existence and uniqueness of (4) are satisfied [5].

The validity of Statement 2 follows from certain simple topological considerations and from the following properties of $c_{12}^{\pm}$:

$$\mp c_{12}^{\pm} \geqslant 0, \qquad c_{12}^{\pm}(r, z_0/2 - z) = -c_{12}^{\pm}(r, z + z_0/2)$$

$$|\lim_{r\to 0} c_{12}^-| < \infty\ |c_{12}^+|\,|_{r=r_0} < \infty, \qquad z \in [0, z_0/2]$$

$$c_{12}^+\big|_{r=0} = -c_{12}^-\big|_{r=0}, \qquad \lim_{r\to r_0} c_{12}^- = 0$$

The validity of Statement 3 follows from the continuity of $c_1^{\pm}$ in D [5] and the following properties of $c_1^{\pm}$:

$$|c_1^-| \leqslant \text{const}\,(r_0 - r)^{-1/2} \text{ в } d_0$$

$$|\lim_{r\to 0} c_1^+| < \infty, \qquad \lim_{r\to r_0}(c_1^- q_r^-) = 0$$

Applying to q_z all operations carried out on q_r, we obtain for $q_z^{\pm}$ the equations

$$c_{21}^{\pm}\partial q_z^{\pm}/\partial r + \partial q_z^{\pm}/\partial z + c_2^{\pm} q_z^{\pm} = 2\pi B \tag{10}$$

where

$$c_{21}^{\pm} = \int_{\Delta^{\pm}} \psi_z^{\pm} \sin\theta \cos\varphi\, d\Omega$$

$$c_2^{\pm} = \frac{\partial c_{21}^{\pm}}{\partial r} - \frac{1}{r}\int_{\Delta^{\pm}} \frac{\partial \psi_z^{\pm}}{\partial \varphi} \sin\theta \sin\varphi\, d\Omega + K\int_{\Delta^{\pm}} \psi_z^{\pm}\, d\Omega$$

$$q_z^{\pm} = \int_{\Delta^{\pm}} I \cos d\Omega$$

$$\Delta^+ = \{\theta \in (0, \pi/2),\ \varphi \in (0, 2\pi)\},\ \Delta^- = \{\theta \in (\pi/2, \pi),\ \varphi \in (0, 2\pi)\}$$

Characteristic directions are defined by the equations

$$dr^{\pm}/dr = c_{12}^{\pm} \tag{11}$$

with boundary conditions

$$q_z^+|_{z=0} = q_z^-|_{z=z_0} = 0 \tag{12}$$

System (10) with boundary conditions (12) is, similarly to problem (3), (5), solvable in D. We call the process of determination of coefficients $c_{i,j}$ and c_i $(i, j = 1, 2)$ "averaging". If K and B are not constant but fairly smooth, Statements 1–3 are valid in the absence of external radiation sources.

Having thus solved the transport equation (1) for all points (r, z) and in all directions inside the cylinder, we can determine coefficients c_{ij} and c_i in Eqs. (3) and (10) and then, using these, determine $q_r^{\pm}$ and $q_z^{\pm}$ independently of transport equations. On the assumption that c_{ij} and c_i vary only slightly with the variation of K and B (with time or in various versions of the stationary problem) it is possible to consider these, within certain limits, as constant at every point and calculate $q_r^{\pm}$ and $q_z^{\pm}$ by (3) and (10) for the changed (within certain limits) K and B. Substituting $q_r^{\pm}$ and $q_z^{\pm}$ determined by the averaged equations of transport into the energy equation which is solved together with other equations of gasdynamics, we can determine the temperature and density at subsequent instants of time and, using the latter, calculate K and B. Then, solving again (1), calculate c_{ij} and c_i and recalculate $q_r^{\pm}$ and $q_z^{\pm}$ by (3) and (10) with certain mean coefficients c_{ij}^* and c_i^* (mean of c_{ij} and c_i at two consecutive instants of averaging).

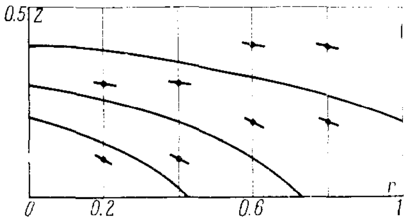

Fig. 2

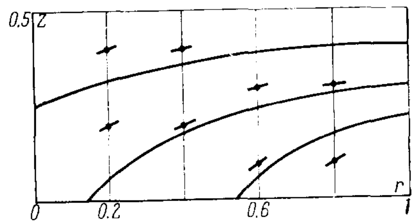

Fig. 3

The direction fields for (4) and (11) are shown in Figs. 2–4 in the case of a uniformly heated cylinder ($K = 1, r_0 = z_0 = 1$). Figs. 2, 3 and 4 show directions related to fields c_{12}^+, c_{12}^- and c_{21}^+, respectively. Since $c_{12}^{\pm}$ is symmetric about the straight line $z = z_0/2$, only regions of $z \leqslant z_0 / 2$ are shown in Figs. 2 and 3 and, owing to the validity of the relationship $c_{21}^{\pm}(r, z) = - c_{21}^-(r, z_0 - z)$ only c_{21}^+ is shown in Fig. 4.

Taking into account the form of coefficients c_{ij}

$$c_{ij} = \int_{\Omega_i} I g_{ij}\, d\Omega \Big/ \int_{\Omega_i} I f_{ij}\, d\Omega \qquad (13)$$

where Ω_i is the related hemisphere and g_{ij} and f_{ij} are functions of φ and θ, we find that inside the cylinder, with the eexception of points at distance $\sim 1 / r_0$ and $1 / z_0$ (from the cylinder end-face and wall, respectively),

$$c_{ij} \approx \int_{\Omega_i} g_{ij}\, d\Omega \Big/ \int_{\Omega_i} f_{ij}\, d\Omega$$

$$\text{for } K r_0 \gg 1, \qquad K z_0 \gg 1 \qquad (14)$$

$$c_{ij} \approx \int_{\Omega_i} s g_{ij}\, d\Omega \Big/ \int_{\Omega_i} s f_{ij}\, d\Omega$$

$$\text{for } K r_0 \ll 1, \qquad K z_0 \ll 1 \qquad (15)$$

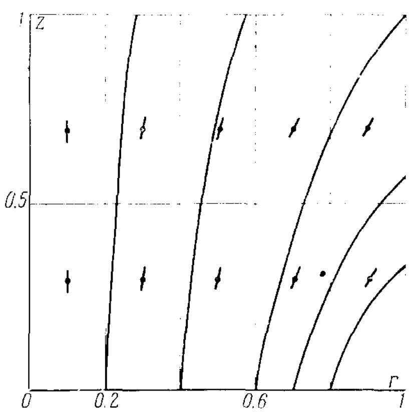

Fig. 4

Formula (14) shows that the field of directions remains virtually unchanged in each case. Substituting explicit expressions of functions g_{ij} and f_{ij} into (14), we find that for optically thick spaces inside the cylinder, except at its edges, $c_{ij} \approx 0$ and $i \neq j$

Calculations carried out for various values of K show a "weak" dependence of the direction field on the optical thickness of space, hence it is possible to expect that averaging can be made also by frequencies, i.e. to obtain, as in [4], averaged equations for the integral density of the radiation stream.

The direction fields derived here for a cylinder of particular dimensions can be used for cylinders of other dimensions, provided that the following conditions:

$$B^* (r, z) = B (r\beta, z\beta), \quad K^* (r, z) = \beta K (r\beta, r\beta)$$
$$\beta = z_0 / z_* = r_0 / r_* \tag{16}$$

are satisfied. In these formulas B^* and K^* relate to a cylinder of radius r_* and hight z_*, and B and K to a cylinder of radius r_0 and height z_0. If conditions (16) are satisfied, then it is possible to show with the use of (1) and (13) that the relationship

$$c_{ij}^* (r, z) = c_{ij} (r\beta, z\beta)$$

is satisfied.

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